JOURNAL OF
COMPUTATIONAL AND
APPLIED MATHEMATICS

ELSEVIER Journal of Computational and Applied Mathematics 68 (1996) 69–78

A noncommutative discrete hypergroup associated with q-disk polynomials

Paul G.A. Floris*

Department of Mathematics and Computer Science, Leiden University, P.O. Box 9512, 2300 RA Leiden, The Netherlands

Received 21 November 1994; revised 31 May 1995

Abstract

Starting from the addition formula for q-disk polynomials we prove nonnegativity of the linearization coefficients for these polynomials, which allows us to realize a DJS-hypergroup structure on the space $\mathbb{Z}_+^2$. This hypergroup is discrete and noncommutative.

Keywords: q-Disk polynomials; Linearization coefficients; DJS-hypergroup

AMS classification: 17B37; 33D45; 33D55; 33D80; 43A62

0. Introduction

In connection with a given system of orthogonal polynomials $\{p_n\}$ it is of great interest to know if there exist positive measures $\mu_{x,y}(z)$ and nonnegative numbers $c_{k,l}(m)$ such that

$$p_n(x)p_n(y) = \int p_n(z)\,\mathrm{d}\mu_{x,y}(z) \tag{0.1}$$

and

$$p_k(x)p_l(x) = \sum_m c_{k,l}(m)\,p_m(x). \tag{0.2}$$

The first formula, called product formula, gives rise to a positive convolution structure and the second formula, called linearization formula, to a dual positive convolution structure associated

* E-mail: floris@wi.leidenuniv.nl.

SSDI 0377-0427(95)00256-1

with these orthogonal polynomials. It is a quite classical result that such positivity results hold for Gegenbauer polynomials $P_n^{(\alpha,\alpha)}$ $(\alpha \geqslant -\frac{1}{2})$. Around 1970, these results were also proved for more general Jacobi polynomials $P_n^{(\alpha,\beta)}$ with (α,β) in a set containing $\{(\alpha,\beta)\in\mathbb{R}^2 \mid \alpha \geqslant \beta \geqslant -\frac{1}{2}\}$ (see [3, 4]). In case such polynomials have an interpretation as spherical functions on a compact symmetric space G/K, these positivity results and associated convolution structures follow immediately from analysis on the space of K-biinvariant functions on G. In connection with this, see also the survey paper by Gasper [5]. In the seventies, the essential aspects of such zonal analysis on groups were abstracted into the concept of a DJS-hypergroup by work of Dunkl, Jewett and Spector. This made it possible for example to conclude that the positivity results in the two product formulas for Jacobi polynomials with quite general parameters α, β give rise to two associated hypergroups, one discrete and one continuous, and dual to each other. For an elaborate exposition of the theory of hypergroups we refer the reader to the book by Bloom and Heyer [1]. For connections with orthogonal polynomials see also Lasser [11].

A formula like (0.1) can often be given in an explicit way. Then it may also be possible to extend it, using Carlson's theorem, by analytic continuation from some discrete set of parameter values to a more general set. However, in (0.2) the coefficients are often not explicitly known. Thus the analytic continuation method will not work, and one must look for an alternative way of proving positivity for the more general parameter set. One way of achieving this uses an addition formula for the polynomials p_n with general parameters, obtained from the discrete case by a continuation argument (see the method described in [7]).

In the theory of quantum groups it is a natural question to ask whether one can obtain results similar to the ones we have in the classical situation. In his paper, Koornwinder [10] showed that it is possible to associate a discrete hypergroup with the "double coset space" of a Gel'fand pair of compact quantum groups, although the construction is somewhat more involved than the classical one. But as in the classical situation the basis ingredient is positivity of linearization coefficients for the related spherical functions. It is perhaps good to note that the hypergroups arising in this way need not be commutative.

The aim of this paper is to give an example of such a noncommutative discrete hypergroup, which is associated with q-disk polynomials. These are polynomials $R_{l,m}^{(\alpha)}$ in two noncommuting variables which are expressed through little q-Jacobi polynomials and that appear, for the value $\alpha = n-2$, as zonal spherical functions on a quantum analogue of the homogeneous space $U(n)/U(n-1)$. This fact was first proved in [12] (see also [2]). In our previous preprint [2], we proved an addition formula for these q-disk polynomials. It is this addition formula that will allow us to prove positivity of linearization coefficients in a manner similar to [7], and to construct from it a DJS-hypergroup following [10]. We remark that for "group" values $\alpha = n-2 \in \mathbb{Z}_+$ this was done by Koornwinder [9, 10].

The paper is organized as follows. In Section 1, we briefly recall the definition of q-disk polynomials and some of their properties. Furthermore, we will state the addition formula which they satisfy. Section 2 merely deals with the proof of positivity, or rather nonnegativity, of the linearization coefficients. The proof resembles the way of reasoning in [7]. Finally, in Section 3 we explicitly construct the noncommutative discrete hypergroup related to the q-disk polynomials.

We conclude this section by fixing the notation and recalling some well-known facts. In all that follows we will keep $0 < q < 1$ fixed.

Recall the definition of the little q-Jacobi polynomials:

$$p_m(x;\, a, b;\, q) = {}_2\varphi_1\left[\begin{matrix} q^{-m},\, abq^{m+1} \\ aq \end{matrix};\, q,\, qx\right] = \sum_{k=0}^{m} \frac{(q^{-m};\, q)_k (abq^{m+1};\, q)_k}{(aq;\, q)_k (q;\, q)_k} (qx)^k.$$

If $0 < aq < 1$ and $bq < 1$, they satisfy the orthogonality

$$\sum_{k=0}^{\infty} \frac{(bq;\, q)_k}{(q;\, q)_k} (aq)^k (p_l p_m)(q^k;\, a, b;\, q) = \delta_{lm} \frac{(q, bq;\, q)_l (aq)^l (1 - abq)(abq^2;\, q)_\infty}{(aq, abq;\, q)_l (1 - abq^{2l+1})(aq;\, q)_\infty}.$$

Here

$$(a;\, q)_\infty = \prod_{j=0}^{\infty} (1 - q^j a), \qquad (a;\, q)_\beta = \frac{(a;\, q)_\infty}{(aq^\beta;\, q)_\infty}$$

are q-shifted factorials ($\beta \in \mathbb{C}$). In particular, if we let

$$P_m^{(\alpha, \beta)}(x : q) := p_m(x;\, q^\alpha, q^\beta;\, q) \quad (\alpha, \beta > -1)$$

then the orthogonality reads

$$\int_0^1 P_l^{(\alpha, \beta)}(x : q) P_m^{(\alpha, \beta)}(x : q) x^\alpha \frac{(qx;\, q)_\infty}{(q^{\beta+1} x;\, q)_\infty} \, \mathrm{d}_q x$$

$$= \delta_{lm} \frac{(1 - q) q^{m(\alpha+1)}}{1 - q^{\alpha+\beta+2m+1}} \frac{(q;\, q)_m (q;\, q)_{\beta+m}}{(q^{\alpha+1};\, q)_m (q^{\alpha+1};\, q)_{\beta+m}}.$$

Here we used Jackson's q-integral

$$\int_0^c f(x)\, d_q x = c(1 - q) \sum_{k=0}^{\infty} f(cq^k) q^k.$$

Finally, write $\mathbb{Z}_+$ for the nonnegative integers

$$\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$$

and put

$$l \wedge m = \min(l, m).$$

1. q-Disk polynomials and their addition formula

Suppose we are given the complex unital $*$-algebra $\mathscr{X}$ generated by the elements z and z^*, subject to the relation

$$z^* z = q^2 z z^* + 1 - q^2 \tag{1.1}$$

and with $*$-structure $(z)^* = z^*$. It is not hard to show that $\mathscr{X}$ has as a linear basis the set of monomials $\{z^k(z^*)^l \colon k, l \in \mathbb{Z}_+\}$. On this algebra we define the *q-disk polynomials* $R_{l,m}^{(\alpha)}(z, z^*; q^2)$ for $\alpha > -1$ and $l, m \in \mathbb{Z}_+$ as follows:

$$R_{l,m}^{(\alpha)}(z, z^*; q^2) := \begin{cases} z^{l-m} P_m^{(\alpha, l-m)}(1 - zz^* : q^2) & (l \geqslant m) \\ P_l^{(\alpha, m-l)}(1 - zz^* : q^2)(z^*)^{m-l} & (l \leqslant m) \end{cases} \tag{1.2}$$

(see also [12, Theorem 4.7], [9, Section 11]). The reason for using q^2, rather than q, in the defining relation of $\mathscr{X}$ and as a base for the q-disk polynomials is mainly for easier reference to [2]. Note that

$$R_{l,m}^{(\alpha)}(z, z^*; q^2)^* = R_{m,l}^{(\alpha)}(z, z^*; q^2). \tag{1.3}$$

It is easily seen that one has $R_{l,m}^{(\alpha)}(z, z^*; q^2) = \sum_{i=0}^{l \wedge m} c_i z^{l-i}(z^*)^{m-i}$ with $c_0 \neq 0$. The orthogonality of these polynomials can be expressed through a double integral:

$$\begin{aligned} &\frac{1}{2\pi} \int_0^1 \int_0^{2\pi} ((R_{l,m}^{(\alpha)})^* R_{l',m'}^{(\alpha)})(\mathrm{e}^{\mathrm{i}\theta} z, \mathrm{e}^{-\mathrm{i}\theta} z^*; q^2)\, \mathrm{d}\theta\, (1 - zz^*)^\alpha \mathrm{d}_{q^2}(1 - zz^*) \\ &\quad = \delta_{ll'}\, \delta_{mm'}\, \frac{1 - q^2}{1 - q^{2(\alpha+1)}}\, c_{l,m}^{(\alpha)}, \end{aligned} \tag{1.4}$$

where

$$c_{l,m}^{(\alpha)} = \frac{(1 - q^{2(\alpha+1)}) q^{2m(\alpha+1)}}{1 - q^{2(\alpha+l+m+1)}}\, \frac{(q^2; q^2)_l\, (q^2; q^2)_m}{(q^{2(\alpha+1)}; q^2)_l\, (q^{2(\alpha+1)}; q^2)_m}. \tag{1.5}$$

Note that (1.4) is well-defined, since after integrating with respect to θ one obtains a polynomial which is invariant under the transformation $z \mapsto \mathrm{e}^{\mathrm{i}\theta} z$, $z^* \mapsto \mathrm{e}^{-\mathrm{i}\theta} z^*$, and hence is a polynomial in the single variable $1 - zz^*$.

The same orthogonality can be achieved using the linear functional $h_{(\alpha)} \colon \mathscr{X} \to \mathbb{C}$ ($\alpha > -1$) defined as

$$h_{(\alpha)}(z^k (z^*)^l) := \delta_{kl}\, q^{2k(\alpha+1)}\, \frac{(q^2; q^2)_k}{(q^{2(\alpha+2)}; q^2)_k}$$

and satisfying $h_{(\alpha)}(p^*) = \overline{h_{(\alpha)}(p)}$ for all $p \in \mathscr{X}$. Then

$$h_{(\alpha)}(R_{l,m}^{(\alpha)}(z, z^*; q^2)^*\, R_{l',m'}^{(\alpha)}(z, z^*; q^2)) = \delta_{ll'}\, \delta_{mm'}\, c_{l,m}^{(\alpha)}. \tag{1.6}$$

It follows that the $R_{l,m}^{(\alpha)}(z, z^*; q^2)$ ($l, m \in \mathbb{Z}_+$) form an orthogonal basis for $\mathscr{X}$ with respect to the inner product defined by $h_{(\alpha)}$. For more details we refer the reader to [2].

In [2, Theorem 3.5.8] we proved the following *addition formula* for these q-disk polynomials:

Theorem 1. *Suppose we are given the abstract complex $*$-algebras $\mathscr{X}$ and $\mathscr{Y}$ with generators X_1, X_2, X_1^*, X_2^* and Y_1, Y_2, Y_1^*, Y_2^* respectively, relations*

$$\begin{aligned} &X_1 X_2 = q X_2 X_1, \qquad X_1^* X_2 = q X_2 X_1^*, \\ &X_2^* X_2 = q^2 X_2 X_2^* + (1 - q^2), \qquad X_1^* X_1 = q^2 X_1 X_1^* + (1 - q^2)(1 - X_2 X_2^*), \end{aligned} \tag{1.7}$$

$$Y_1 Y_2 = qY_2Y_1, \qquad Y_1^* Y_2 = qY_2Y_1^*,$$
$$Y_1Y_1^* = Y_1^*Y_1, \qquad 1 = Y_1Y_1^* + Y_2Y_2^* = q^2Y_1^*Y_1 + Y_2^*Y_2$$

and $*$-structures

$$\begin{aligned} (X_1)^* &= X_1^*, \qquad (X_2)^* = X_2^*, \\ (Y_1)^* &= Y_1^*, \qquad (Y_2)^* = Y_2^*. \end{aligned} \tag{1.8}$$

Then, for arbitrary $\alpha > 0$ and arbitrary $l, m \in \mathbb{Z}_+$ we have the following addition formula for the q-disk polynomials:

$$\begin{aligned} &R_{l,m}^{(\alpha)}(-qX_1\otimes Y_1^* + X_2\otimes Y_2,\ -qX_1^*\otimes Y_1 + X_2^*\otimes Y_2^*; q^2) \\ &= \sum_{r=0}^{l}\sum_{s=0}^{m} c_{l,m;r,s}^{(\alpha)} R_{l-r,m-s}^{(\alpha+r+s)}(X_2, X_2^*; q^2) R_{r,s}^{(\alpha-1)}(X_1, X_1^*, 1 - X_2X_2^*; q^2) \\ &\qquad \otimes (-q)^{r-s} R_{l-r,m-s}^{(\alpha+r+s)}(Y_2, Y_2^*; q^2) Y_1^s (Y_1^*)^r. \end{aligned} \tag{1.9}$$

Here

$$c_{l,m;r,s}^{(\alpha)} = \frac{1 - q^{2(\alpha+r+s+1)}}{1 - q^{2(\alpha+1)}} \frac{c_{l,m}^{(\alpha)}}{c_{l-r,m-s}^{(\alpha+r+s)} c_{r,s}^{(\alpha-1)}},$$

(*cf.* (1.5)), *and*

$$R_{l,m}^{(\alpha)}(A, B, C; q) := \begin{cases} C^m A^{l-m} P_m^{(\alpha, l-m)}\left(\dfrac{C - AB}{C} : q\right) & (l \geqslant m) \\ C^l P_l^{(\alpha, m-l)}\left(\dfrac{C - AB}{C} : q\right) B^{m-l} & (l \leqslant m). \end{cases} \tag{1.10}$$

Note that with the notation of (1.10) we have that $R_{l,m}^{(\alpha)}(A, B, 1; q) = R_{l,m}^{(\alpha)}(A, B; q)$. Define the map $\tilde{h}_{(\alpha)} : \mathscr{X} \to \mathscr{X}$ as

$$\tilde{h}_{(\alpha)}(p(X_2, X_2^*)X_1^k(X_1^*)^l) := \delta_{kl}\, p(X_2, X_2^*)(1 - X_2X_2^*)^k q^{2k\alpha} \frac{(q^2; q^2)_k}{(q^{2(\alpha+1)}; q^2)_k}. \tag{1.11}$$

Here $p(X_2, X_2^*)$ is any (ordered) polynomial in X_2, X_2^* and k, l are in $\mathbb{Z}_+$.

Lemma 2. *Let $p_1(X_2, X_2^*)$ and $p_2(X_2, X_2^*)$ be arbitrary ordered polynomials in X_2, X_2^* and let $p_3(X_1, X_1^*, 1 - X_2X_2^*)$ be any ordered polynomial in $X_1, X_1^*, 1 - X_2X_2^*$, homogeneous of degree k, where we put $\deg(X_1) = \deg(X_1^*) = \frac{1}{2}$ and $\deg(1 - X_2X_2^*) = 1$. Then*

$$\begin{aligned} &\tilde{h}_{(\alpha)}(p_1(X_2, X_2^*)p_3(X_1, X_1^*, 1 - X_2X_2^*)p_2(X_2, X_2^*)) \\ &= h_{(\alpha-1)}(p_3(X_1, X_1^*, 1))p_1(X_2, X_2^*)(1 - X_2X_2^*)^k p_2(X_2, X_2^*). \end{aligned}$$

Proof. In view of (1.7) we can write

$$p_3(X_1, X_1^*, 1 - X_2X_2^*) = \sum_{i=0}^{k} \sum_{j=0}^{k-i} c_{ij}(1 - X_2X_2^*)^i X_1^j (X_1^*)^{-j+2(k-i)}.$$

The assertion now easily follows when one uses the first two relations of (1.7) and the relations $X_2(1 - X_2X_2^*) = q^{-2}(1 - X_2X_2^*)X_2$ and $X_2^*(1 - X_2X_2^*) = q^2(1 - X_2X_2^*)X_2^*$, which are immediate from (1.7). □

Combining Lemma 2 with (1.3), (1.6) and (1.5) we obtain as a consequence

Corollary 3. *Let $\tilde{h}_{(\alpha)}$ be defined as in* (1.11). *Then there holds*

$$\begin{aligned}\tilde{h}_{(\alpha)}(&R^{(\alpha+i+j)}_{l-i,m-j}(X_2, X_2^*; q^2) R^{(\alpha-1)}_{i,j}(X_1, X_1^*, 1 - X_2X_2^*; q^2) \\ &\times R^{(\alpha-1)}_{p,r}(X_1, X_1^*, 1 - X_2X_2^*; q^2)^* R^{(\alpha+p+r)}_{l'-p,m'-r}(X_2, X_2^*; q^2)^*) \\ &= \delta_{ip}\delta_{jr} c^{(\alpha-1)}_{j,i} R^{(\alpha+i+j)}_{l-i,m-j}(X_2, X_2^*; q^2)(1 - X_2X_2^*)^{i+j} R^{(\alpha+p+r)}_{l'-p,m'-r}(X_2, X_2^*; q^2)^*.\end{aligned}$$

2. Positivity of linearization coefficients

Since the $R^{(\alpha)}_{l,m}(z, z^*; q^2)$ $(l, m \in \mathbb{Z}_+)$ form a linear basis for $\mathscr{X}$, we have the following expansion in $\mathscr{X}$:

$$R^{(\alpha)}_{l,m}(z, z^*; q^2) R^{(\alpha)}_{l',m'}(z, z^*; q^2)^* = \sum_{l'',m'' \in \mathbb{Z}_+} a(l, m; l', m'; l'', m'') R^{(\alpha)}_{l'',m''}(z, z^*; q^2). \tag{2.1}$$

Here only finitely many of the coefficients $a(l, m; l' m'; l'', m'')$ are nonzero if we fix l, m, l', m'. In fact the sum ranges over those values of l'', m'' such that $l'' - m'' = l - m - (l' - m')$ and $l'' + m'' \leqslant l + m + l' + m'$. The $a(l, m; l', m'; l'', m'')$ depend on α and are called *linearization coefficients.*

Theorem 4. *For all $\alpha > 0$ and all possible choices of (l, m), (l', m'), $(l'', m'') \in \mathbb{Z}_+^2$ the linearization coefficients in* (2.1) *are nonnegative*:

$$a(l, m; l', m'; l'', m'') \geqslant 0.$$

Proof. First note that both the pair X_2, X_2^* and the pair Y_2, Y_2^* satisfy (1.1). Let $\Omega = -qX_1 \otimes Y_1^* + X_2 \otimes Y_2$. It is straightforward to verify that $\Omega^*\Omega = q^2\Omega\Omega^* + 1 - q^2$. This means that we have an identity similar to (2.1) but with z, z^* replaced by Ω, Ω^*:

$$R^{(\alpha)}_{l,m}(\Omega, \Omega^*; q^2) R^{(\alpha)}_{l',m'}(\Omega, \Omega^*; q^2)^* = \sum_{l'',m'' \in \mathbb{Z}_+} a(l, m; l', m'; l'', m'') R^{(\alpha)}_{l'',m''}(\Omega, \Omega^*; q^2). \tag{2.2}$$

Now substitute (1.9) into the right-hand side as well as for both factors in the left-hand side of (2.2) and apply $\tilde{h}_{(\alpha)}\otimes id$ to this (cf. (1.11)). Using Lemma 2 and (1.6) for the right-hand side and Corollary 3 for the left-hand side, we obtain after a straightforward calculation

$$\begin{aligned}
&\sum_{i=0}^{l\wedge l'} \sum_{j=0}^{m\wedge m'} c^{(\alpha)}_{l,m;\,i,j}\, c^{(\alpha)}_{l',m';\,i,j}\, c^{(\alpha-1)}_{j,i}\, q^{2(i-j)} \\
&\qquad \times R^{(\alpha+i+j)}_{l-i,m-j}(X_2,X_2^*;q^2)(1-X_2X_2^*)^{i+j} R^{(\alpha+i+j)}_{l'-i,m'-j}(X_2,X_2^*;q^2)^* \\
&\qquad \otimes R^{(\alpha+i+j)}_{l-i,m-j}(Y_2,Y_2^*;q^2)(1-Y_2Y_2^*)^{i+j} R^{(\alpha+i+j)}_{l'-i,m'-j}(Y_2,Y_2^*;q^2)^* \\
&\quad = \sum_{l'',m''} a(l,m;l',m';l'',m'')\, R^{(\alpha)}_{l'',m''}(X_2,X_2^*;q^2)\otimes R^{(\alpha)}_{l'',m''}(Y_2,Y_2^*;q^2).
\end{aligned} \tag{2.3}$$

Write σ for the $*$-algebra anti-automorphism $\sigma:\mathscr{Y}\to\mathscr{Y}$ which interchanges Y_2 and Y_2^* and fixes Y_1 and Y_1^*. Note that

$$\sigma(R^{(\alpha)}_{r,s}(Y_2,Y_2^*;q^2)) = R^{(\alpha)}_{r,s}(Y_2,Y_2^*;q^2)^*.$$

Letting $id\otimes\sigma$ act on (2.3) yields

$$\begin{aligned}
&\sum_{l'',m''} a(l,m;l',m';l'',m'')\, R^{(\alpha)}_{l'',m''}(X_2,X_2^*;q^2)\otimes R^{(\alpha)}_{l'',m''}(Y_2,Y_2^*;q^2)^* \\
&\quad = \sum_{i=0}^{l\wedge l'} \sum_{j=0}^{m\wedge m'} c^{(\alpha)}_{l,m;\,i,j}\, c^{(\alpha)}_{l',m';\,i,j}\, c^{(\alpha-1)}_{j,i}\, q^{2(i-j)} \\
&\qquad \times R^{(\alpha+i+j)}_{l-i,m-j}(X_2,X_2^*;q^2)(1-X_2X_2^*)^{i+j} R^{(\alpha+i+j)}_{l'-i,m'-j}(X_2,X_2^*;q^2)^* \\
&\qquad \otimes R^{(\alpha+i+j)}_{l'-i,m'-j}(Y_2,Y_2^*;q^2)(1-Y_2Y_2^*)^{i+j} R^{(\alpha+i+j)}_{l-i,m-j}(Y_2,Y_2^*;q^2)^*.
\end{aligned}$$

Finally, multiply from the left with $R^{(\alpha)}_{l_0'',m_0''}(X_2,X_2^*;\ q^2)^*\otimes 1$ and from the right with $1\otimes R^{(\alpha)}_{l_0'',m_0''}(Y_2,Y_2^*;q^2)$ and evaluate $h_{(\alpha)}\otimes h_{(\alpha)}$ on the result. By virtue of (1.6) we wind up with

$$\begin{aligned}
&a(l,m;l',m';l_0'',m_0'')\,(c^{(\alpha)}_{l_0'',m_0''})^2 \\
&\quad = \sum_{i=0}^{l\wedge l'} \sum_{j=0}^{m\wedge m'} c^{(\alpha)}_{l,m;\,i,j}\, c^{(\alpha)}_{l',m';\,i,j}\, c^{(\alpha-1)}_{j,i}\, q^{2(i-j)}\, |\, h_{(\alpha)}(R^{(\alpha)}_{l_0'',m_0''}(X_2,X_2^*;q^2)^* \\
&\qquad \times R^{(\alpha+i+j)}_{l-i,m-j}(X_2,X_2^*;q^2)(1-X_2X_2^*)^{i+j} R^{(\alpha+i+j)}_{l'-i,m'-j}(X_2,X_2^*;q^2)^*)\,|^2
\end{aligned}$$

since $h_{(\alpha)}$ satisfies $h_{(\alpha)}(p^*)=\overline{h_{(\alpha)}(p)}$. Since $0<q<1$, this will imply that $a(l,m;l',m';l'',m'')\geqslant 0$ for all choices of (l,m), (l',m') and (l'',m''). □

Remark. Considering the case where $l=m$ we thus obtain nonnegativity for the linearization coefficients of the little q-Jacobi polynomials $P^{(\alpha,0)}_l(x:q^2)$ with $\alpha>0$.

Remark. For the values $\alpha=n-2\in\mathbb{Z}_+$, corresponding to the "group" case, this result was obtained in Koornwinder [9].

3. A discrete hypergroup structure associated with q-disk polynomials

In this section we construct a so-called DJS-hypergroup from the linearization formula (2.1), following Koornwinder [10]. The crucial ingredient for this was proved in Theorem 4. But first we define the proper setting.

Let K be a locally compact Hausdorff topological space. Write $M(K)$ for the space of all complex regular Borel measures of K and $M^1(K)$ for the subset of all probability measures. For $x \in K$ we denote by δ_x the corresponding point measure: $\delta_x(\{x\}) = 1$ (so $\delta_x \in M^1(K)$). Assume that in addition there exist

(a) *convolution*: a map $K \times K \to M^1(K)$, $(x, y) \to \delta_x \star \delta_y$, continuous in the weak topology with respect to $C_c(K)$;

(b) *involution*: an involutive homeomorphism $K \to K$, $x \to \bar{x}$;

(c) *unit element*: a distinguished element $e \in K$.

Upon identifying x with δ_x, the map in (a) extends uniquely to a continuous bilinear map $M(K) \times M(K) \to M(K)$, $(\mu, \nu) \to \mu \star \nu$. The involution of (b) induces an involution $\mu \to \mu^*$ on $M(K)$ as follows: $\mu^*(E) = \mu(\bar{E})$ (E is a Borel subset of K).

Definition. The quadruple $(K, \star, ^{-}, e)$ is called a *DJS-hypergroup* if for all $x, y, z \in K$ the following conditions are met:

(1) $\delta_x \star (\delta_y \star \delta_z) = (\delta_x \star \delta_y) \star \delta_z$,

(2) $\operatorname{supp}(\delta_x \star \delta_y)$ is compact,

(3) $(\delta_x \star \delta_y)^* = \delta_{\bar{y}} \star \delta_{\bar{x}}$,

(4) $\delta_e \star \delta_x = \delta_x = \delta_x \star \delta_e$,

(5) $e \in \operatorname{supp}(\delta_{\bar{x}} \star \delta_y)$ if and only if $x = y$,

(6) the map of $K \times K$ to the space of nonvoid compact subsets of K given by $(x, y) \to \operatorname{supp}(\delta_x \star \delta_y)$ is continuous. Here the target space has the topology as defined in [6, Section 2.5].

The hypergroup is called *commutative* if $\delta_x \star \delta_y = \delta_y \star \delta_x$ for all $x, y \in K$, otherwise it is called *noncommutative*.

Theorem 5. *Put $K = \mathbb{Z}_+^2$, endowed with the discrete topology. For (l, m), (l', m') and $(l'', m'') \in K$, define*

$$(\delta_{(l,m)} \star \delta_{(l',m')})(\{(l'', m'')\}) = a(l, m; m', l'; l'', m'')$$

with $a(l, m; l', m'; l'', m'')$ as in (2.1) (note the permutation in the pair (l', m')). As an involution on K take $(l, m)^- = (m, l)$. Furthermore write $e = (0, 0)$. Then the quadruple $(K, \star, ^-, e)$ forms a noncommutative discrete DJS-hypergroup.

Proof. We have to verify (a), (1)–(6). Let us abbreviate $R_{l,m}^{(\alpha)}(z, z^*; q^2)$ by $R_{l,m}^{(\alpha)}$.

(a): The convolution is continuous since we have given K the discrete topology. Define the multiplicative linear functional $\varepsilon : \mathscr{Z} \to \mathbb{C}$ by $\varepsilon(z) = 1 = \varepsilon(z^*)$ (so in fact ε is identically 1 on $\mathscr{Z}$). If we now apply ε to both sides of (2.1) we get

$$1 = \sum_{l'', m''} a(l, m; l', m'; l'', m'')$$

which implies that $\delta_{(l,m)} \star \delta_{(l',m')} \in M^1(K)$.

(1): Use the associativity $R^{(\alpha)}_{l,m}(R^{(\alpha)}_{l',m'} R^{(\alpha)}_{l'',m''}) = (R^{(\alpha)}_{l,m} R^{(\alpha)}_{l',m'}) R^{(\alpha)}_{l'',m''}$ together with (1.3) and (2.1) to find

$$\begin{aligned} R^{(\alpha)}_{l,m}(R^{(\alpha)}_{l',m'} R^{(\alpha)}_{l'',m''}) = R^{(\alpha)}_{l,m}(R^{(\alpha)}_{l',m'}(R^{(\alpha)}_{m'',l''})^*) &= \sum_{r,s} a(l', m'; m'', l''; r, s) R^{(\alpha)}_{l,m}(R^{(\alpha)}_{s,r})^* \\ &= \sum_{r,s,u,v} a(l', m'; m'', l''; r, s) a(l, m; s, r; u, v) R^{(\alpha)}_{u,v} \\ &= (R^{(\alpha)}_{l,m} R^{(\alpha)}_{l',m'}) R^{(\alpha)}_{l'',m''} \\ &= \sum_{r,s} a(l, m; m', l'; r, s) R^{(\alpha)}_{r,s}(R^{(\alpha)}_{m'',l''})^* \\ &= \sum_{r,s,u,v} a(l, m; m', l'; r, s) a(r, s; m'', l'': u, v) R^{(\alpha)}_{u,v}. \end{aligned}$$

So

$$\begin{aligned} &\sum_{r,s,u,v} a(l', m'; m'', l''; r, s) a(l, m; s, r; u, v) \\ &\quad = \sum_{r,s,u,v} a(l, m; m', l'; r, s) a(r, s; m'', l''; u, v), \end{aligned}$$

and hence $\delta_{(l,m)} \star (\delta_{(l',m')} \star \delta_{(l'',m'')}) = (\delta_{(l,m)} \star \delta_{(l',m')}) \star \delta_{(l'',m'')}$.

(2): Since only finitely many of the elements $a(l, m; l', m'; l'', m'')$ are nonzero when (l, m), (l', m') are fixed, the support of $\delta_{(l,m)} \star \delta_{(l',m')}$ is compact.

(3): We know that $(R^{(\alpha)}_{l,m}(R^{(\alpha)}_{m',l'})^*)^* = R^{(\alpha)}_{m',l'}(R^{(\alpha)}_{l,m})^*$, which gives

$$\sum_{l'',m''} a(l, m; m', l'; l'', m'') R^{(\alpha)}_{m'',l''} = \sum_{l'',m''} a(m', l'; l, m; l'', m'') R^{(\alpha)}_{l'',m''},$$

whence

$$a(l, m; m', l'; m'', l'') = a(m', l'; l, m; l'', m'').$$

From this it follows that $(\delta_{(l,m)} \star \delta_{(l',m')})^* = \delta_{(m',l')} \star \delta_{(m,l)} = \delta_{(l',m')^-} \star \delta_{(l,m)^-}$.

(4): Since $R^{(\alpha)}_{0,0} = 1$ we get that

$$a(l, m; 0, 0; l'', m'') = \delta_{l,l''} \delta_{m,m''} = a(0, 0; m, l; l'', m'').$$

Consequently $\delta_{(l,m)} \star \delta_{(0,0)} = \delta_{(l,m)} = \delta_{(0,0)} \star \delta_{(l,m)}$.

(5): Note that, in view of (1.6) and (1.3),

$$\begin{aligned} \delta_{ll'} \delta_{mm'} c^{(\alpha)}_{l,m} &= h_{(\alpha)}((R^{(\alpha)}_{l,m})^* R^{(\alpha)}_{l',m'}) \\ &= h_{(\alpha)}(R^{(\alpha)}_{m,l}(R^{(\alpha)}_{m',l'})^*) \\ &= \sum_{l'',m''} a(m, l; m', l'; l'', m'') h_{(\alpha)}(R^{(\alpha)}_{l'',m''}) \\ &= a(m, l; m', l'; 0, 0). \end{aligned}$$

So $e = (0, 0) \in \mathrm{supp}(\delta_{(l,m)^-} \star \delta_{(l',m')}) = \mathrm{supp}(\delta_{(m,l)} \star \delta_{(l',m')})$ if and only if one has $a(m, l;\ m', l';\ 0, 0) \neq 0$, which, by the above, is true only in case $(l, m) = (l', m')$.

(6): Obvious since K has discrete topology.

Finally, this hypergroup is noncommutative since $\delta_{(l,m)} \star \delta_{(l'm')} = \delta_{(l',m')} \star \delta_{(l,m)}$ would imply that $R^{(\alpha)}_{l,m} R^{(\alpha)}_{l',m'} = R^{(\alpha)}_{l',m'} R^{(\alpha)}_{l,m}$ and in general this is not the case (e.g. $R^{(\alpha)}_{1,0} R^{(\alpha)}_{0,1} \neq R^{(\alpha)}_{0,1} R^{(\alpha)}_{1,0}$). □

Remark. We conclude with the observation that K contains a commutative (even hermitian) subhypergroup H, namely $H = \{(l, l) : l \in \mathbb{Z}_+\} \simeq \mathbb{Z}_+$. It corresponds to the little q-Jacobi polynomials $P^{(\alpha,0)}_l(x : q^2)$ $(\alpha > 0)$.

Acknowledgements

The author thanks Professor Tom H. Koornwinder for his useful comments and suggestions.

References

[1] W.R. Bloom and H. Heyer, *Harmonic Analysis of Probability Measures on Hypergroups* (De Gruyter, Berlin, 1994).
[2] P.G.A. Floris, Addition formula for q-disk polynomials, Report W94-24, Mathematical Institute, University of Leiden, 1994; also: *Compositio Math.*, to appear.
[3] G. Gasper, Linearization of the product of Jacobi polynomials. I, *Canad. J. Math.* **22** (1970) 171–175.
[4] G. Gasper, Linearization of the product of Jacobi polynomials. II, *Canad. J. Math.* **22** (1970) 582–593.
[5] G. Gasper, Positivity and special functions, in: R. Askey, Ed., *Theory and Application of Special Functions* (Academic Press, New York, 1975) 375–433.
[6] R.I. Jewett, Spaces with an abstract convolution of measures, *Adv. in Math.* **18** (1975) 1–101.
[7] T.H. Koornwinder, Positivity proofs for linearization and connection coefficients of orthogonal polynomials satisfying an addition formula, *J. London Math. Soc.* **18** (2) (1978) 101–114.
[8] T.H. Koornwinder, Orthogonal polynomials in connection with quantum groups, in: P. Nevai, Ed., *Orthogonal Polynomials: Theory and Practice*, NATO ASI Series, Vol. 294 (Kluwer Academic Publishers, Dordrecht, 1990) 257–292.
[9] T.H. Koornwinder, Positive convolution structures associated with quantum groups, in: H. Heyer, Ed., *Probability Measures on Groups X* (Plenum, New York, 1991) 249–268.
[10] T.H. Koornwinder, Discrete hypergroups associated with compact quantum Gelfand pairs, in: W.C. Connett, M.-O. Gebuhrer and A.L. Schwartz, Eds., *Applications of Hypergroups and Related Measure Algebras*, Contemporary Math. (Amer. Math. Soc., Providence, RI, 1995) 213–235.
[11] R. Lasser, Orthogonal polynomials and hypergroups, *Rend. Math.* **2** (7) (1983) 185–209.
[12] M. Noumi, H. Yamada and K. Mimachi, Finite dimensional representations of the quantum group $GL_q(n, \mathbb{C})$ and the zonal spherical functions on $U_q(n-1)/U_q(n)$, *Japan. J. Math.* **19** (1) (1993) 31–80.